COMPUTATION OF THE INDEX OF LINEAR ELLIPTIC OPERATORS IN UNBOUNDED CYLINDERS

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Abstract. Linear Fredholm elliptic operators in unbounded cylinders are considered. A formula for the index of the operator, i.e. the difference between the dimensions of the kernel and cokernel is obtained. The key point is the splitting of the operator obtained by using spectral projectors of the Laplace operator in the (bounded) transverse domain. This, together with a continuation method, allows the reduction to one dimensional systems for which the index may be computed explicitly.

1. Introduction

We consider a linear elliptic operator of the form

$$Lu = a(x)\Delta u + \sum_{i=1}^{n} b_i(x)\frac{\partial u}{\partial x_i} + c(x)u. \tag{1.1}$$

Here $x = (x_1, ..., x_n) \in \Omega$, where $\Omega \subset I\!R^n$ is an unbounded cylinder: $\Omega = I\!R \times \Omega'$, and Ω' is a bounded domain in $I\!R^{n-1}$. Thus the axis of the cylinder is parallel to the x_1-direction, and we denote by $x' = (x_2, ..., x_n) \in \Omega'$ the transversal variable. The unknown function is vector-valued: $u = (u_1, ..., u_p)$, and $a(x)$, $b_i(x)$, $c(x)$ are $p \times p$ matrices. The matrix $a(x)$ is supposed to be positive definite,

$$(a(x)u, u) \geq k(u, u)$$

for some $k > 0$, and any $u \in I\!R^p$, $x \in \bar{\Omega}$. Δ denotes the Laplace operator, understood componentwise.

The boundary S of the cylinder Ω is supposed to be of class $C^{2+\delta}$ for some fixed $0 < \delta < 1$. We will use homogeneous Dirichlet boundary conditions:

$$u = 0 \quad \text{on} \quad S = I\!R \times \partial\Omega'.$$

For any integer l, by $C^{l+\delta}(\bar{\Omega}, I\!R^p)$ we denote the Banach space of functions which are bounded and continuous in $\bar{\Omega}$, together with all partial derivatives up to order l, and such that the partial derivatives of order l are uniformly Hölder continuous with exponent δ.

We suppose that all the coefficients of the operator L belong to $C^{\delta}(\bar{\Omega}, I\!R)$, and have limits as $x_1 \to \pm\infty$:

$$a^{\pm} = \lim_{x_1 \to \pm\infty} a(x), \;\; b_i^{\pm} = \lim_{x_1 \to \pm\infty} b_i(x), \;\; c^{\pm} = \lim_{x_1 \to \pm\infty} c(x).$$

Here $a^{\pm}$, $b_i^{\pm}$, and $c^{\pm}$ are constant matrices.

The operator L is considered as acting between the Banach space E of functions u in the Hölder space $C^{2+\delta}(\bar{\Omega}, I\!R^p)$ satisfying the boundary condition $u = 0$ on S, and the space $C^{\delta}(\bar{\Omega}, I\!R^p)$. The corresponding norms will be denoted by $|\cdot|_{2+\delta,\Omega}$ and $|\cdot|_{\delta,\Omega}$ respectively.

If we consider the operator L in a bounded domain in $I\!R^n$ with a smooth boundary, then the operator is Fredholm with index zero. Here we understand the index as the difference between the dimensions α of the kernel and β of the cokernel of the operator:

$$ind(L) = \alpha(L) - \beta(L).$$

As is well-known the index can be different from zero for elliptic operators of a more general form [14, 15, 16] or for domains with corners [3]. For the case of elliptic operators on manifolds and an introduction to Atiyah-Singer theory we refer the reader to [2]. Elliptic operators in unbounded domains have been the object of much attention in the past twenty years. In the case of scalar equations, let us mention the article [8], where asymptotically homogeneous elliptic operators on noncompact manifolds are studied. The computation of the index is based on the study of the mapping properties of the fundamental solution, a technique introduced in [12]. We refer the reader to [9] [10] for a computation of the index of elliptic operators in $I\!R^n$ under similar assumptions, and to [1] for similar results in Hölder spaces. A common feature of all these studies is that the elliptic operator A under investigation is of the form $A = A_\infty + B$, where the constant-coefficient operator A_∞ is homogeneous, and the coefficients of B tend to 0 at infinity, with an appropriate rate. This implies that the kernel and cokernel of the operator A (considered between appropriate weighted spaces) consist of polynomials, and have finite dimensions which can be explicitly computed.

The problem which we study here is of a different nature because the limit values of the coefficients at $x_1 = +\infty$ and $x_1 = -\infty$ may be different, and the limit operators $L^\pm$ are not necessarily homogeneous.

We show that the index of operators of the form (1.1) in smooth unbounded domains can be different from zero, and we compute its value. Let us illustrate the situation on the following example. We consider the following simplest equation on the real axis:

$$u'(t) = r(t)u(t) + f(t).$$

Here $u(t)$ is bounded together with its first derivative, $r(t)$ is continuous and has limits at infinity, $r_\pm = r(\pm\infty)$, $f(t)$ is continuous and bounded. We assume, for simplicity, that $r(t) = r_+$ for t sufficiently large, $r(t) = r_-$ for $-t$ sufficiently large. (This assumption is not necessary.)

The solution of this equation can be written explicitly:

$$u(t) = e^{\int_0^t r(s)ds} \left(\int_0^t e^{-\int_0^y r(s)ds} f(y)dy + u(0) \right).$$

If $r_\pm \neq 0$, then the problem is Fredholm and there are four cases depending on the signs of r_+ and r_-. For each of them we can easily find the number of linearly independent bounded solutions of the homogeneous equation and the number of solvability conditions of the nonhomogeneous equation. In particular, if $r_+ > 0$ and $r_- < 0$, then the solvability condition is

$$\int_{-\infty}^{\infty} e^{-\int_0^y r(s)ds} f(y)dy = 0.$$

The index of the problem is given by the following formula:

$$ind = s_+ + s_- - 1,$$

where s_+ is the number of solutions of the equation $du/dt = r_+u$ which are bounded near $+\infty$, s_- is the number of solutions of the equation $du/dt = r_-u$ which are bounded near $-\infty$. Of course, s_+ and s_- can be easily connected to the signs of r_+ and r_-.

For systems of equations, solutions cannot be found explicitly. However, it is still possible to compute the index and it has a similar form:

$$ind = s_+ + s_- - p.$$

Here s_+ and s_- are the numbers of bounded solutions of the corresponding limiting equations at $+\infty$ and $-\infty$, and p is the dimension of the system. In other words, s_+ is the number of eigenvalues of the matrix r_+ with negative real part, s_- is the number of eigenvalues of the matrix r_- with positive real part. As we will see in Section 3, both the dimension of the kernel of the corresponding operator and the codimension of its image can change under small deformations. However, their difference, i.e. the index of the operator remains the same under deformations in the class of Fredholm operators. This is a well known fact in the theory of Fredholm operators. However, it is interesting to see explicitly how some extra terms appear in the α and β characteristics, but do not change their difference.

To compute the index for second order operators, we reduce them to first order operators. Finally, to study elliptic operators in cylinders, we use spectral projectors corresponding to the Laplace operator in the section of the cylinder and reduce the multidimensional problem to one-dimensional problems along the axis of the cylinder.

An interesting question connected with the computation of the index concerns homotopy classes of Fredholm operators. We show that there are different homotopy classes of Fredholm elliptic operators with the same index. This can already be seen on elementary examples (Section 3.5).

One of the essential points in the computation of the index is to use the fact that it does not change under deformations of the operator in the class of Fredholm operators.

The contents of the paper are as follows. We discuss conditions under which the operators considered in Hölder spaces are Fredholm in Section 2. We compute the index for ordinary differential operators in Section 3, and for elliptic partial differential operators in Section 4.

In what follows, for any operator L we denote the null-space by $N(L)$, the range by $R(L)$, the $\alpha-$ characteristic of L is the dimension of $N(L)$ and the $\beta-$ characteristic is the codimension of $R(L)$.

2. Fredholm operators

We recall that an operator is said to be Fredholm if its kernel and cokernel are finite dimensional. We now give conditions under which elliptic operators are Fredholm and recall some properties of Fredholm operators which will be used in the sequel.

Consider the operators L^+ and L^- acting between E and $C^\delta(\bar{\Omega}, I\!R^p)$, defined by:

$$L^\pm u = a^\pm \Delta u + \sum_{i=1}^{n} b_i^\pm \frac{\partial u}{\partial x_i} + c^\pm u, \tag{2.1}$$

and the problem

$$L_\xi^\pm v = \lambda v, \quad v_{|\partial G} = 0, \tag{2.2}$$

where

$$L_\xi^\pm v = a^\pm \Delta' v + \sum_{k=2}^n b_k^\pm \frac{\partial v}{\partial x_k} + c^\pm v - \xi^2 a^\pm v + i\xi b_1^\pm v,$$

$$\Delta' v = \sum_{k=2}^n \frac{\partial^2 v}{\partial x_k^2}.$$

Here the operators $L_\xi^\pm$ are considered as acting between $C^{2+\delta}(\bar\Omega', \mathbb{C}^p)$ and $C^\delta(\bar\Omega', \mathbb{C}^p)$. The Problem (2.2) is obtained from the corresponding eigenvalue problem for the operators $L^\pm$, by applying (formally) the Fourier transform with respect to the x_1-variable. We use the following condition:

Condition 1. *For all real ξ, the problem (2.2) does not have the eigenvalue $\lambda = 0$.*

In [18] we proved the following theorem, under more general assumptions on the coefficients of the operator (see also [11, 13]):

Theorem 2.1. *Condition 1 is a necessary and sufficient condition for the operator L to be normally solvable with a finite α-characteristic.*

Suppose that there exists a continuous deformation L_τ such that for each τ Condition 1 is satisfied. If for some value τ_0 the β-characteristic is infinite, then it is infinite for all τ [5]. Hence if it is finite for some τ, then it is finite for all τ and the operator is Fredholm also for all τ. Then its index does not change. Thus we can use the following approach to compute the index. We construct a continuous deformation of the operator L to some other operator L_0 with a finite β-characteristic. If this deformation is fulfilled in the class of operators satisfying Condition 1, then $ind(L) = ind(L_0)$. In particular, we can change the operator in such a way that the limiting values of its coefficients at $\pm\infty$ do not change.

3. Ordinary differential operators on the real line

3.1. Notations

Hereafter, for any integer l the notation $C_b^l(\mathbb{R}, \mathbb{R}^p)$ denote the space of functions which are defined on $\mathbb{R}$, take their values in $\mathbb{R}^p$, and whose derivatives of all orders up to l are continuous and bounded. In particular, the space $C_b^0(\mathbb{R}, \mathbb{R}^p)$ is a Banach space for the uniform norm. Let M be a continuous matrix-valued function defined on $\mathbb{R}$, having limits at infinity:

$$M(t) \to A_\pm \quad \text{as} \quad t \to \pm\infty.$$

On $C_b^0(\mathbb{R}, \mathbb{R}^p)$ we consider the operator T defined by:

$$u \mapsto u' + Mu,$$

with domain $D(T) = C_b^1(\mathbb{R}, \mathbb{R}^p)$.

Together with T we consider the operator $\tilde{T}$ which has the same domain, and is associated to some continuous matrix-valued function A (in place of M) such that:

$$A(t) = A_- \quad \text{for} \quad t \le -1, \quad A_+ \quad \text{for} \quad 1 \le t. \tag{3.1}$$

We will denote by $E_\pm$ the spectral projector associated to the set of all eigenvalues of $A_\pm$ having positive real part. We shall repeatedly use the fact that $E_\pm$ commutes with any function of $A_\pm$ whenever the latter can be defined.

We will assume the following condition on the spectrum of $A_\pm$:

$$\sigma(A_\pm) \cap i\mathbb{R} = \emptyset. \tag{3.2}$$

We wish to show that under this assumption the operator $\tilde{T}$ is Fredholm, and to compute its index in terms of the projectors $E_\pm$ (see [4, 15, 6] for related results on the half-line). The main tool we use is the variation of constants formula.

3.2. Representation of the solution

Let $f \in C_b^0(\mathbb{R}, \mathbb{R}^p)$ be given, and let $u \in D(\tilde{T})$ be a solution to the equation

$$\frac{du}{dt} + Au = f. \tag{3.3}$$

Let us denote by $Y = Y(t)$ the fundamental matrix associated to $\tilde{T}$, defined by :

$$\frac{dY}{dt} + A(t)Y = 0, \quad Y(-1) = I.$$

We remark that Y can be written explicitly:

$$\text{for} \quad t \le -1: \quad Y(t) = e^{-(t+1)A_-}, \tag{3.4}$$

$$\text{for} \quad t \ge 1: \quad Y(t) = e^{-(t-1)A_+}\Lambda, \tag{3.5}$$

where $\Lambda = Y(1)$ is an invertible $p \times p$ matrix.

Therefore, in the range $|t| \ge 1$, the solution u is given by:

$$\begin{aligned} \text{for} \quad t \le -1: \quad u(t) &= e^{-(t+1)A_-}u(-1) - \int_t^{-1} e^{-(t-s)A_-} f(s)ds, \\ \text{for} \quad 1 \le t: \quad u(t) &= e^{-(t-1)A_+}\Lambda u(-1) + e^{-(t-1)A_+}\Lambda \int_{-1}^{1} Y(s)^{-1} f(s)ds \\ &\quad + e^{-(t-1)A_+} \int_1^t e^{(s-1)A_+} f(s)ds. \end{aligned} \tag{3.6}$$

Let us first consider the case $t > 1$. The formula defining $u(t)$ yields:

$$\begin{aligned} e^{(t-1)A_+}(I - E_+)u(t) &= (I - E_+)\Lambda u(-1) + (I - E_+)\Lambda \int_{-1}^{1} Y(s)^{-1} f(s)ds \\ &\quad + \int_1^t e^{(s-1)A_+}(I - E_+) f(s)ds. \end{aligned} \tag{3.7}$$

The l.h.s. of (3.7) tends to 0 as $t \to +\infty$, thus we get a first compatibility relation:

$$\begin{aligned}(I - E_+)\Lambda u(-1) &= -(I - E_+)\Lambda \int_{-1}^{1} Y(s)^{-1} f(s)\, ds \\ &\quad - \int_{1}^{\infty} e^{(s-1)A_+}(I - E_+) f(s)\, ds. \end{aligned} \tag{3.8}$$

Using this relation we can now express $u(t)$ for $t > 1$ as follows:

$$\begin{aligned} u(t) &= e^{-(t-1)A_+} E_+ \Lambda [u(-1) + \int_{-1}^{1} Y(s)^{-1} f(s)\, ds] \\ &\quad + \int_{1}^{t} e^{-(t-s)A_+} E_+ f(s)\, ds - \int_{t}^{\infty} e^{-(t-s)A_+}(I - E_+) f(s)\, ds. \end{aligned} \tag{3.9}$$

Note that all integrals appearing in this expression are automatically finite.
Similarly for $t < -1$ we have the relation

$$e^{(t+1)A_-} E_- u(t) = E_- u(-1) - \int_{t}^{-1} e^{(s+1)A_-} E_- f(s)\, ds, \tag{3.10}$$

which yields

$$E_- u(-1) = \int_{-\infty}^{-1} e^{(s+1)A_-} E_- f(s) ds. \tag{3.11}$$

This in turn gives the following expression of $u(t)$ for $t < -1$:

$$\begin{aligned} u(t) &= e^{-(t+1)A_-}(I - E_-) u(-1) - \int_{t}^{-1} e^{-(t-s)A_-}(I - E_-) f(s)\, ds \\ &\quad + \int_{-\infty}^{t} e^{-(t-s)A_-} E_- f(s) ds. \end{aligned} \tag{3.12}$$

To summarize, we have established:

Proposition 3.1 *Assume that (3.2) is satisfied. If the function $f \in C_b^0(\mathbb{R}, \mathbb{R}^p)$ is such that $f = \bar{T}u$ for some $u \in D(\bar{T})$, then any such u is given by (3.9) for $t > 1$ and (3.12) for $t < -1$, for some $u(-1) \in \mathbb{R}^p$ satisfying (3.8) and (3.11).*

3.3. The kernel of $\bar{T}$.

If $\bar{T}u = 0$, then by setting $f = 0$ in (3.8) and (3.11) we obtain:

$$u(-1) \in N(E_-), \quad \Lambda u(-1) \in N(I - E_+).$$

Noting that Λ is an isomorphism, we may conclude that:

$$\alpha(\tilde{T}) = \dim[N(E_-) \cap \Lambda^{-1} N(I - E_+)].$$

3.4. The image of $\tilde{T}$.

For the sake of convenience, let us introduce the following notation for $f \in C^0_b(I\!R, I\!R^p)$:

$$\begin{aligned} I_1(f) &= (I - E_+)\Lambda \int_{-1}^{1} Y(s)^{-1} f(s) ds \\ &\quad + \int_{1}^{\infty} e^{(s-1)A_+}(I - E_+) f(s)\, ds, \\ I_2(f) &= \int_{-\infty}^{-1} e^{(s+1)A_-} E_- f(s) ds. \end{aligned}$$

Note that $(I - E_+) I_1(f) = I_1(f)$, and that $E_- I_2(f) = I_2(f)$. If $f \in C^0_b(I\!R, I\!R^p)$ is in the range of $\tilde{T}$, then from (3.8), (3.11) we obtain for some $u(-1) \in I\!R^p$ the condition:

$$E_- u(-1) = I_2(f), \quad (I - E_+)\Lambda u(-1) = -I_1(f). \tag{3.13}$$

We first give an equivalent form of this condition:

Lemma 3.2 *The last condition is equivalent to the following requirement:*

$$\Lambda I_2(f) + I_1(f) \in N(I - E_+) + \Lambda N(E_-). \tag{3.14}$$

To prove necessity, assuming (3.13), we note that

$$(I - E_+)(I_1(f) + \Lambda u(-1)) = 0, \quad E_-(I_2(f) - u(-1)) = 0,$$

thus for some v_1, v_2 we must have:

$$u(-1) = I_2(f) + (I - E_-)v_2, \quad \Lambda u(-1) = -I_1(f) - E_+ v_1.$$

Multiplying the first equality by Λ and subtracting, we obtain (3.14). To prove sufficiency, suppose that

$$I_1(f) + \Lambda I_2(f) = -\Lambda(I - E_-)v_2 - E_+ v_1.$$

Define $u(-1)$ by

$$u(-1) = I_2(f) + (I - E_-)v_2. \tag{3.15}$$

Then we have

$$\begin{aligned} \Lambda u(-1) &= \Lambda I_2(f) + \Lambda(I - E_-)v_2 \\ &= -I_1(f) - E_+ v_1. \end{aligned} \tag{3.16}$$

Thus (3.15) and (3.16) respectively yield

$$E_-(u(-1) - I_2(f)) = 0,$$
$$(I - E_+)(\Lambda u(-1) + I_1(f)) = 0.$$

Then (3.13) is satisfied, and this proves the lemma.

We have seen that if $f \in C_b^0(\mathbb{R}, \mathbb{R}^p)$ is in the range of $\bar{T}$, then (3.14) is satisfied. Conversely if (3.14) is satisfied, then for some $u(-1) \in \mathbb{R}^p$, (3.13) is satisfied. We may then define u by (3.9) for $t > 1$ and (3.12) for $t < -1$. To show that $f \in R(\bar{T})$, it remains to show that we can define u in the interval $[-1, 1]$ so as to obtain a global solution. The Cauchy problem with initial data at $t = -1$ has a unique solution which satisfies:

$$u(1) = \Lambda[u(-1) + \int_{-1}^{1} Y(s)^{-1} f(s)\, ds].$$

Thus from (3.9), we see that we can obtain a global solution if we have continuity at $t = 1$, i.e. if the following relation:

$$E_+\Lambda[u(-1) + \int_{-1}^{1} Y(s)^{-1} f(s)\, ds] - \int_{1}^{\infty} e^{(s-1)A_+}(I - E_+) f(s)\, ds$$
$$= \Lambda[u(-1) + \int_{-1}^{1} Y(s)^{-1} f(s)\, ds]$$

holds true. This relation is an immediate consequence of the second equality in (3.13), thus we obtain that $f \in R(\bar{T})$.

Using condition (3.14) we can now compute the $\beta-$ characteristic of $\bar{T}$. Define a linear map

$$\Phi : C_b^0(\mathbb{R}, \mathbb{R}^p) \longrightarrow \frac{\mathbb{R}^p}{N(I - E_+) + \Lambda N(E_-)}$$

by assigning to each $f \in C_b^0(\mathbb{R}, \mathbb{R}^p)$ the coset of $I_1(f) + \Lambda I_2(f)$, denoted by $[I_1(f) + \Lambda I_2(f)]$. From what we have seen, we have

$$R(\bar{T}) = N(\Phi).$$

Thus

$$\beta(\bar{T}) = \mathrm{codim} R(\bar{T}) = \dim R(\Phi).$$

We now show that Φ is surjective. Pick $x \in \mathbb{R}^p$, we need to find $f \in C_b^0(\mathbb{R}, \mathbb{R}^p)$ such that:

$$I_1(f) + \Lambda I_2(f) - x \in N(I - E_+) + \Lambda N(E_-).$$

If we were working with the space of (not necessarily continuous) bounded functions we could take

$$f(t) = -A_+ x \quad \forall t > 1, \quad f(t) = 0 \quad \forall t < 1.$$

This function f satisfies:

$$I_1(f) + \Lambda I_2(f) - x = -E_+ x,$$

and therefore $\Phi(f) = [x]$. However this f is not continuous, thus using this idea we set

$$\begin{aligned} f_k(t) &= 0 \quad \text{for} \quad t \leq 1, \\ & k(1-t)A_+x \quad \text{for} \quad 1 \leq t \leq 1+\frac{1}{k}, \\ & -A_+x \quad \text{for} \quad t \geq 1+\frac{1}{k}. \end{aligned}$$

It is easy to compute that

$$I_1(f_k) + \Lambda I_2(f_k) - x = -E_+x + x_k,$$

where

$$x_k \to 0 \quad \text{as} \quad k \to \infty.$$

Since the projection map $[\cdot]$ is continuous we have

$$\Phi(f_k) \to [x],$$

and since the image of Φ (being finite-dimensional) is closed we conclude again that it contains the point $[x]$.

Summarizing, we have shown the following:

$$\begin{aligned} \beta(\tilde{T}) &= p - \dim[N(I-E_+) + \Lambda N(E_-)] \\ &= p - \dim[\Lambda^{-1}N(I-E_+) + N(E_-)]. \end{aligned}$$

Note that to obtain the last equality we used the fact that Λ is an isomorphism.

Obviously, the operator T is Fredholm iff the operator $\tilde{T}$ is Fredholm. Using the remark at the end of Section 2, we conclude that they have the same index.

Finally we obtain the index of the Fredholm operator T:

$$\begin{aligned} \text{ind}(T) &= \dim\Lambda^{-1}N(I-E_+) + \dim N(E_-) - p \\ &= \dim N(I-E_+) + \dim N(E_-) - p \end{aligned}$$

Therefore we have shown:

Theorem 3.3 *If condition (3.2) is satisfied then T is a Fredholm operator, and its index is given by:*

$$\text{ind}(T) = \dim N(I-E_+) - \dim N(I-E_-).$$

In particular, we see that the index is zero if $E_- = E_+$.

We note that the formula for index can be also formulated in terms of the number of linearly independent bounded solutions at infinity. Namely, the index is equal to the sum of the numbers of bounded linearly independent solutions at minus and plus infinity, minus the dimension of the system p. The particular case of zero index is considered in [6].

3.5. Second order operators

We consider the operator

$$L : C^{2+\delta}(\mathbb{R}, \mathbb{R}^p) \longrightarrow C^{\delta}(\mathbb{R}, \mathbb{R}^p) \tag{3.17}$$

defined by the expression:

$$u \longmapsto a(x)u^{''} + b(x)u^{'} + c(x)u.$$

Here $a(x)$, $b(x)$, and $c(x)$ are smooth $p \times p$ matrices having respectively the limits $a^{\pm}, b^{\pm}, c^{\pm}$ as $x \to \pm\infty$. We will use the results of the previous subsection to show that under some appropriate condition L is a Fredholm operator, and to compute its index.

In a previous paper we showed (see theorem 2.1 in [18] for a more general statement) the following result:

Lemma 3.4 *Assume that*

$$\forall \xi \in \mathbb{R} : \quad T^{\pm}(\xi) = -a^{\pm}\xi^2 + b^{\pm}i\xi + c^{\pm} \quad \text{is an invertible matrix.} \tag{3.18}$$

then the operator L is normally solvable.

We first rewrite the system $Lu = f$ as a first order system, and consider the first order ordinary differential operators M and T:

$$M : C^{2+\delta}(\mathbb{R}, \mathbb{R}^p) \times C^{1+\delta}(\mathbb{R}, \mathbb{R}^p) \longrightarrow C^{1+\delta}(\mathbb{R}, \mathbb{R}^p) \times C^{\delta}(\mathbb{R}, \mathbb{R}^p), \tag{3.19}$$

$$T : C^1_b(\mathbb{R}, \mathbb{R}^{2p}) \longrightarrow C^0_b(\mathbb{R}, \mathbb{R}^{2p}), \tag{3.20}$$

associated to the expression:

$$(u, q) \longmapsto (u, q)' + A(u, q).$$

Here A denotes the matrix:

$$A(x) = \begin{pmatrix} 0 & -I_p \\ a^{-1}c & a^{-1}b \end{pmatrix},$$

and I_p denotes the $p \times p$ identity matrix. The result in section 3.4 (Th.3.3) shows that if condition (3.2) is satisfied, then T is a Fredholm operator, and we have a formula for its index in terms of the limit matrices $A^{\pm}$. We remark that condition (3.2) is equivalent to condition (3.18). We now proceed to show that under this condition, the operators M and L are also Fredholm, and have the same index as T.

Lemma 3.5 *If T is Fredholm, then M also is, and has the same index.*

Lemma 3.6 *Assume that M is Fredholm, and that L is normally solvable. Then L also is Fredholm, and has the same index as M.*

Combining these three lemmas we immediately obtain the desired result:

Theorem 3.7 *Assume that condition (3.18) is satisfied. Then L is a Fredholm operator, and its index is given by*

$$ind(L) = \kappa^+ - \kappa^-,$$

where κ^+ and κ^- are, respectively, the number of eigenvalues of the matrices A^+ and A^- with positive real part.

Proof of Lemma 3.5

Let us consider the system of equations

$$u' = p + f_1, \quad p' = a^{-1}(bp + cu + f_2).$$

If the solution of the homogeneous problem belongs to $C^1(I\!R, I\!R^p)$, then $u' = p \in C^{1+\delta}(I\!R, I\!R^p)$ (since the coefficients are Hölder continuous), and $u(x) \in C^{2+\delta}(I\!R, I\!R^p)$. Hence the dimension of the kernel remains the same. Now let $f_1 \in C^{1+\delta}(I\!R, I\!R^p)$, $f_2 \in C^{\delta}(I\!R, I\!R^p)$. If (f_1, f_2) satisfies the solvability conditions in C^1, then there is a solution $u \in C^1(I\!R, I\!R^p), p \in C^1(I\!R, I\!R^p)$. As above, it follows that $u \in C^{2+\delta}(I\!R, I\!R^p)$ and $p \in C^{1+\delta}(I\!R, I\!R^p)$. If (f_1, f_2) does not satisfy the solvability conditions in C^1, then obviously there are no solutions of this problem with $u \in C^{2+\delta}(I\!R, I\!R^p)$ and $p \in C^{1+\delta}(I\!R, I\!R^p)$. Hence the solvability conditions and their number remain the same.

Proof of Lemma 3.6

We first consider the kernel of L. If $u \in N(L)$, then we have $(u, u') \in N(M)$. Conversely if $(u, q) \in N(M)$, then $q = u'$, and $u \in N(L)$. This shows that the map $u \mapsto (u, u')$ realizes an isomorphism between $N(L)$ and $N(M)$, thus we have $\alpha(L) = \alpha(M)$. Let us now consider the range of L. For fixed $(f_1, f_2) \in C^{1+\delta}(I\!R, I\!R^p) \times C^{\delta}(I\!R, I\!R^p)$, from $M(u, q) = (f_1, f_2)$ it follows that $Lu = af_1' + bf_1 + af_2$. For convenience we define the map

$$\pi : C^{1+\delta}(I\!R, I\!R^p) \times C^{\delta}(I\!R, I\!R^p) \longrightarrow C^{\delta}(I\!R, I\!R^p) \tag{3.21}$$

$$(f_1, f_2) \longmapsto af_1' + bf_1 + af_2. \tag{3.22}$$

Then clearly we have

$$\pi(R(M)) = R(L).$$

By assumption $R(M)$ has finite codimension in $C^{1+\delta}(I\!R, I\!R^p) \times C^{\delta}(I\!R, I\!R^p)$, and $R(L)$ is closed in $C^{\delta}(I\!R, I\!R^p)$. Our aim is to show that the codimensions of these two spaces are equal. Define $N_1 = \beta(M) < \infty$, and $N_2 = \beta(L) \leq \infty$, and denote by $\{\Phi_i, i = 1, ..., N_1\}$ a basis of the annihilator of $R(M)$. Each Φ_i is a bounded linear functional on $C^{1+\delta}(I\!R, I\!R^p) \times C^{\delta}(I\!R, I\!R^p)$. Similarly, denote by $\{F_k, k = 1, ..., N_2\}$ a basis of the annihilator of $R(L)$. Then for fixed $f = (f_1, f_2) \in C^{1+\delta}(I\!R, I\!R^p) \times C^{\delta}(I\!R, I\!R^p)$ we have the following equivalence:

$$f \in R(M) \Leftrightarrow \Phi_i(f) = 0, i = 1, ..., N_1, \Leftrightarrow \pi f \in R(L) \Leftrightarrow F_k(\pi f) = 0, k = 1, ...N_2.$$

Writing $G_k = F_k \pi$, we thus have:

$$\{f : \Phi_i(f) = 0, i = 1, ..., N_1\} = \{f : G_k(f) = 0, k = 1, ..., N_2\}.$$

The functionals G_k form a family of linearly independent bounded linear functionals on the space $C^{1+\delta}(I\!R, I\!R^p) \times C^{\delta}(I\!R, I\!R^p)$. Thus it follows that $N_1 = N_2$, i.e. the $\beta-$ characteristics of the operators L and M are the same.

Remarks:

- It is easy to verify that $\kappa^{\pm}$ is the number of solutions to the equation

$$det(a^{\pm}\lambda^2 - b^{\pm}\lambda + c^{\pm}) = 0$$

  which have positive real part. As for the case of first order systems of equations, we can say that the index of the operator L is equal to the sum of the numbers of linearly independent bounded solutions of the equation $Lu = 0$ at plus and minus infinity, minus $2p$.

- $\alpha(L)$ and $\beta(L)$ are determined by the projectors E^+ and E^-. These projectors are determined by the matrices A^+ and A^- and remain the same if we consider the operator M as acting from $C^{1+\delta}(I\!R, I\!R^p) \times C^{\delta}(I\!R, I\!R^p)$ or from $C^{\delta}(I\!R, I\!R^p) \times C^{\delta}(I\!R, I\!R^p)$.

As an example we consider the case $p = 1$ where $u(x)$ is a scalar valued function and suppose that $a(x) \equiv 1$, b is a positive constant. If $c^+ \neq 0$, $c^- \neq 0$, then the operator is Fredholm. Its index equals 0 if c^+ and c^- have the same sign, it is 1 for $c^+ > 0$, $c^- < 0$, and -1 for $c^+ < 0$, $c^- > 0$. The essential spectrum of the operators consists of two parabolas on the complex plane. In the case $c^+ < 0$, $c^- < 0$ both of them are completely in the left half-plane. In the case $c^+ > 0$, $c^- > 0$ both of them are partially in the right half-plane having the origine inside the region where the large negative numbers are. Obviously, these two curves cannot be moved to the left half-plane by a continuous deformation such that they do not intersect zero. Hence two elliptic operators with the same index are not necessarily homotopic in the class of Fredholm operators of the same form.

4. Elliptic operators

4.1. Projection on the eigenvalues of the transversal Laplacian

In this section we compute the index of the operator defined in the introduction by (1.1). We assume that $b_k \to 0$, $k = 2, ..., n$ as $x_1 \to \pm\infty$. We consider L as acting between the space

$$E := \{u \in C^{2+\delta}(\bar{\Omega}, I\!R^p) : \quad u_{|S} = 0\}$$

and the space $C^{\delta}(\bar{\Omega}, I\!R^p)$. However to compute this index it will be more convenient to view L as an unbounded operator acting in $C^{\delta}(\bar{\Omega}, I\!R^p)$, with domain $D(L) = E$. We do so from now on. The index is the same, since the kernel and range are unchanged. To compute $ind(L)$

we connect L to the operator $\tilde{L}$ acting in $C^\delta(\bar{\Omega}, I\!R^p)$, with the same domain E, defined by:

$$\tilde{L}u = \bar{a}(x_1)\Delta u + \bar{b}(x_1)\frac{\partial u}{\partial x_1} + \bar{c}(x_1)u. \tag{4.1}$$

For the operator $\tilde{L}$ we take coefficients which only depend on the variable x_1:

$$\bar{a}(x_1) = a^+\psi(x_1) + a^-(1-\psi(x_1)), \;\; \bar{b}(x_1) = b_1^+\psi(x_1) + b_1^-(1-\psi(x_1)), \tag{4.2}$$

$$\bar{c}(x_1) = c^+\psi(x_1) + c^-(1-\psi(x_1)), \tag{4.3}$$

where $\psi(x_1)$ is a sufficiently smooth function, equal to 1 for $x_1 \geq 1$ and to 0 for $x_1 \leq 0$.

We note that during the deformation

$$\tau\tilde{L} + (1-\tau)L, \;\; \tau \in [0,1]$$

Condition 1 is satisfied and the index of the operator does not change (as explained at the end of section 2). Thus instead of the operator (1.1) we will consider the operator (4.1). From now on we omit the tilde and it is this operator which we denote by L.

Consider the (scalar) Laplace operator in the transverse domain:

$$\Delta' : \quad C^\delta(\bar{\Omega}', I\!R) \longrightarrow C^\delta(\bar{\Omega}', I\!R),$$

with domain

$$D(\Delta') := \{u \in C^{2+\delta}(\bar{\Omega}', I\!R) : \; u = 0 \, \text{on} \, \partial\Omega'\}.$$

Since Δ' is formally self-adjoint and Ω' is bounded, the spectrum of this operator consists of a countable sequence of eigenvalues. Let us consider these eigenvalues in the following order (without repetition):

$$\cdots\omega_k < \cdots < \omega_2 < \omega_1 < 0.$$

For each k, the multiplicity m_k of ω_k is finite, and we will denote by $\phi_i^k, i = 1..m_k$ an $L^2(\Omega')$ orthonormal basis of the associated eigenspace:

$$i = 1, ..., m_k : \quad \phi_i^k(x') \in I\!R, \, \Delta'\phi_i^k = \omega_k\phi_i^k \quad \text{in } \Omega', \, \phi_i^k = 0 \quad \text{on } \partial\Omega',$$

Now consider the (vector) Laplacian in the transverse domain:

$$\Delta' : \quad C^\delta(\bar{\Omega}', I\!R^p) \longrightarrow C^\delta(\bar{\Omega}', I\!R^p), \tag{4.4}$$

$$D(\underline{\Delta}') := E' := \{u \in C^{2+\delta}(\bar{\Omega}', I\!R^p) : \; u = 0 \, \text{on } \partial\Omega'\}.$$

The spectrum of this self-adjoint operator is the same as for the scalar Laplacian, but the multiplicity of ω_k is now pm_k. Denote by π_k' the spectral projector associated to the eigenvalue ω_k of the vector Laplacian. Then π_k' is a bounded operator acting in $C^\delta(\bar{\Omega}', I\!R^p)$, and takes its values in E'. We will use the fact that π_k' is continuous for the $C^0(\bar{\Omega}', I\!R^p)$ norm. This is an immediate consequence of the representation of π_k' as a Dunford integral over a bounded contour, see for instance [4]. If $u \in R(\pi_k')$, then each component of u is a linear combination of the functions $\phi_i^k, i = 1..m_k$. Thus we obtain:

$$u(x') = \sum_{i=1}^{m_k} p_i \phi_i^k(x'), \tag{4.5}$$

with $p_i \in I\!R^p$ for each i. From now on let s be a fixed integer. We set

$$P_s' = \sum_{k=1}^{s} \pi_k'.$$

In particular, if Γ is a closed contour in the complex plane containing the first s eigenvalues, for any $v \in C^\delta(\bar{\Omega}', I\!R^p)$ we have

$$P_s' v = \frac{1}{2i\pi} \int_\Gamma (\Delta' - \lambda)^{-1} v d\lambda,$$

and the range of P_s' is equal to the corresponding eigenspace which we denote by E_s'. Finally, we define Q_s' as a bounded operator acting in $C^\delta(\bar{\Omega}', I\!R^p)$ by

$$Q_s' u = u - P_s' u.$$

We have

$$E_s' = P_s'[C^\delta(\bar{\Omega}', I\!R^p)], \;\; C^\delta(\bar{\Omega}', I\!R^p) = E_s' \bigoplus \tilde{E}_s', \quad \text{where} \quad \tilde{E}_s' := Q_s'[C^\delta(\bar{\Omega}', I\!R^p)].$$

Remark 4.1 As we did for π_k', we note that P_s' and Q_s' are continuous for the $C^0(\bar{\Omega}', I\!R^p)$ norm.

Let us set

$$E_s = \{u \in C^\delta(\bar{\Omega}, I\!R^p) : \forall x_1 \in I\!R : \quad u(x_1, \cdot) \in E_s'\},$$
$$\tilde{E}_s = \{u \in C^\delta(\bar{\Omega}, I\!R^p) : \forall x_1 \in I\!R : \quad u(x_1, \cdot) \in \tilde{E}_s'\}.$$

We now define two operators P_s, Q_s on $C^\delta(\bar{\Omega}, I\!R^p)$ by:

$$(P_s u)(x_1, \cdot) = P_s'(u(x_1, \cdot)), \quad (Q_s u)(x_1, \cdot) = Q_s'(u(x_1, \cdot)) \tag{4.6}$$

We first show a preliminary result:

Lemma 4.2
P_s and Q_s are bounded projectors acting in $C^\delta(\bar{\Omega}, I\!R^p)$. Moreover for any $u \in E$ we have:

$$P_s u, Q_s u \in E, \quad \text{and} \quad P_s L u = L P_s u, \; Q_s L u = L Q_s u. \tag{4.7}$$

Proof of Lemma 4.2

We first show that P_s (and therefore Q_s) takes its values in $C^\delta(\bar\Omega, \mathbb{R}^p)$. This amounts to checking the regularity of $P_s u$ with respect to x_1, for $u \in C^\delta(\bar\Omega, \mathbb{R}^p)$. For fixed $x_1 \in \mathbb{R}$, $x' \in \bar\Omega'$, using Remark 4.1 we have:

$$\begin{aligned} |(P_s u)(x_1+h, x') - (P_s u)(x_1, x')| &\leq \|P_s^{'}(u(x_1+h,\cdot)) - P_s^{'}(u(x_1,\cdot))\|_{0,\bar\Omega'} \\ &\leq K\|u(x_1+h,\cdot) - u(x_1,\cdot)\|_{0,\bar\Omega'} \\ &\leq K\|u\|_{\delta,\bar\Omega} h^\delta. \end{aligned} \tag{4.8}$$

Inequality (4.8) shows that P_s, Q_s are bounded operators acting in $C^\delta(\bar\Omega, \mathbb{R}^p)$. Clearly (from the analogous relations for $P_s^{'}, Q_s^{'}$) we have

$$(P_s)^2 = P_s, \quad (Q_s)^2 = Q_s, \quad P_s + Q_s = I.$$

Now since $P_s^{'}$ takes its values in E', it follows that for $u \in C^{2+\delta}(\bar\Omega, \mathbb{R}^p)$, the function $P_s u$ satisfies the boundary condition $P_s u = 0$ on S. To show that $P_s u \in E$, again we only need to check regularity with respect to x_1. If $u \in C^{1+\delta}(\bar\Omega, \mathbb{R}^p)$ we have, as $h \to 0$:

$$\frac{u(x_1+h,\cdot) - u(x_1,\cdot)}{h} \to \frac{\partial u}{\partial x_1}(x_1,\cdot) \quad \text{in} \quad C^0(\bar\Omega', \mathbb{R}^p).$$

Thus, using Remark 4.1:

$$\begin{aligned} \frac{(P_s u)(x_1+h,\cdot) - (P_s u)(x_1,\cdot)}{h} &= P_s^{'}\frac{u(x_1+h,\cdot) - u(x_1,\cdot)}{h} \\ &\to P_s^{'}\frac{\partial u}{\partial x_1}(x_1,\cdot) = (P_s\frac{\partial u}{\partial x_1})(x_1,\cdot) \\ &\quad \text{in} \quad C^0(\bar\Omega', \mathbb{R}^p). \end{aligned}$$

We obtain for any $u \in C^{1+\delta}(\bar\Omega, \mathbb{R}^p)$:

$$\frac{\partial}{\partial x_1}(P_s u) = P_s(\frac{\partial u}{\partial x_1}),$$

thus for $u \in C^{2+\delta}(\bar\Omega, \mathbb{R}^p)$:

$$\frac{\partial^2}{\partial x_1^2}(P_s u) = P_s(\frac{\partial^2 u}{\partial x_1^2}). \tag{4.9}$$

This shows that E is invariant under P_s and Q_s. Clearly $P_s^{'}$ (as a spectral projector of Δ') commutes with Δ', and with multiplication by functions of x_1. Thus (4.7) follows from (4.9), and this completes the proof of the Lemma.

Remark 4.3

$R(Q_s)$ is closed for the topology of uniform convergence on compact subsets of $\bar\Omega$.

Indeed if $f_n \in R(Q_s)$ converges to f uniformly on compact subsets of $\bar\Omega$, then for all x_1, the function $f_n(x_1,\cdot)$ converges to $f(x_1,\cdot)$ uniformly on $\bar{\Omega}'$. Then it follows from Remark 4.1 that $(P_s' f)(x_1,\cdot)=0$, which means that $f\in R(Q_s)$.

To compute the index of L we now define two operators L_1, L_2 by restricting L to $R(P_s)$ and $R(Q_s)$ respectively, and with domains:

$$D(L_1)=D(L)\cap R(P_s), \quad D(L_2)=D(L)\cap R(Q_s).$$

It follows from (4.7) that their ranges are given by:

$$R(L_1)=R(L)\cap R(P_s), \quad R(L_2)=R(L)\cap R(Q_s).$$

This proves that if Condition 1 holds true, then the unbounded operators

$$L_1, L_2: \quad C^\delta(\bar{\Omega}, \mathbb{R}^p) \longrightarrow C^\delta(\bar{\Omega}, \mathbb{R}^p)$$

are normally solvable. By using an elementary lemma on the index of a direct sum (see Appendix 1) we obtain the following result:

Theorem 4.4

Assume the operator L satisfies Condition 1, and define the operators L_1, L_2 as above. If these two operators are Fredholm then L also is, and

$$ind(L)=ind(L_1)+ind(L_2).$$

4.2. Computation of $ind(L_1)$

Defining a map π_k by

$$(\pi_k u)(x_1,\cdot)=\pi_k'(u(x_1,\cdot)),$$

we obtain (as we did with P_s, Q_s) a bounded projector acting in $C^\delta(\bar{\Omega}, \mathbb{R}^p)$. Also, E is invariant under π_k, and on E, the operators L and π_k commute. We can now define an unbounded operator L^k by restricting L to $R(\pi_k)$, with domain $D(L^k)=E\cap R(\pi_k)$. By the same argument as above we have

$$ind(L_1)=\sum_{k=1}^{s} ind(L^k).$$

If $u\in R(\pi_k)$ we have $u(x_1,\cdot)\in R(\pi_k')$ for each x_1, thus in view of (4.5) we obtain:

$$u(x_1,x')=\sum_{i=1}^{m_k} p_i(x_1)\phi_i^k(x'),$$

with $p_i\in C^\delta(\mathbb{R}, \mathbb{R}^p)$ for each i. Then for a given $f\in R(\pi_k)$ written in the form

$$f(x_1,x')=\sum_{i=1}^{m_k} g_i(x_1)\phi_i^k(x'),$$

the equation $L^k u = f$ may be rewritten as:

$$\sum_{i=1}^{m_k}(a(x_1)p_i^{''} + b(x_1)p_i^{'} + (c(x_1) + \omega_k a(x_1))p_i - g_i(x_1))\phi_i^k(x^{'}) = 0.$$

Multiplying by $\phi_i^k(x^{'})$ and integrating over Ω' we obtain the ordinary differential system:

$$ap_i^{''} + bp_i^{'} + (c + \omega_k a)p_i = g_i.$$

The index of this one-dimensional problem (which is given by the results in the previous section), which we denote by ind_k^0 does not depend on i, and we obtain:

$$ind(L^k) = m_k ind_k^0.$$

4.3. Computation of $ind(L_2)$

We now show that if s is sufficiently large, then $ind(L_2) = 0$. We note that for μ large enough, the operator

$$L - \mu : \quad C^\delta(\bar{\Omega}, I\!R^p) \longrightarrow C^\delta(\bar{\Omega}, I\!R^p)$$

with domain E has a bounded inverse defined on all of $C^\delta(\bar{\Omega}, I\!R^p)$ (see Appendix 2), thus has index 0. Therefore it suffices to show that for all $\lambda \geq 0$ the operator

$$L_2 - \lambda : R(Q_s) \rightarrow R(Q_s)$$

is normally solvable and has finite α characteristic (for then we may use again the same method, with $\lambda = \tau\mu$, $\tau \in [0,1]$). We first prove a preliminary lemma:

Lemma 4.5
If s is large enough, then for any λ with $Re(\lambda) \geq 0$, any $j \geq s+1$ and $\xi \in I\!R$:

$$det(-a^\pm \xi^2 + ib^\pm \xi + c^\pm + a^\pm \omega_j - \lambda) \neq 0.$$

Proof of Lemma 4.5
We use the fact that $\omega_j \rightarrow -\infty$ as $j \rightarrow \infty$. Consider an eigenvalue λ of the matrix

$$(\xi^2 - \omega_j)(-a^\pm + \frac{1}{\xi^2 - \omega_j}(ib^\pm \xi + c^\pm)).$$

Noting that

$$||\frac{1}{\xi^2 - \omega_j}(ib^\pm \xi + c^\pm)|| \leq \frac{1}{2\sqrt{-\omega_j}}||b^\pm|| + \frac{||c^\pm||}{-\omega_j},$$

we obtain that any such λ (recall $a^\pm$ is positive definite) has negative real part, provided j is large enough. This proves the claim.

Therefore it only remains to prove the following two propositions:

Proposition 4.6
For all j, ξ, λ, define

$$E_j^{\pm}(\xi,\lambda) = -a^{\pm}\xi^2 + ib^{\pm}\xi + c^{\pm} + a^{\pm}\omega_j - \lambda.$$

If $\lambda \in \mathbb{C}$ is such that

$$\forall \xi \in I\!R,\ j \geq s+1: \quad det(E_j^{\pm}(\xi,\lambda)) \neq 0, \tag{4.10}$$

then the operator

$$L_2 - \lambda:\ R(Q_s) \to R(Q_s)$$

has finite α characteristic.

Proposition 4.7
Under the same assumption on λ, the operator $L_2 - \lambda$ is normally solvable.

Before proving these results we show:

Lemma 4.8
Under the same assumption on λ, the problem

$$a^{\pm}\Delta u + b^{\pm}\frac{\partial u}{\partial x_1} + c^{\pm}u = \lambda u, \quad u_{|S} = 0 \tag{4.11}$$

has no nontrivial solution in $R(Q_s)$.

Proof of Lemma 4.8
Proceeding by contradiction, let us assume that (4.10) holds true, and let $u \in R(Q_s)$ be a nontrivial solution to (4.11).

For fixed $k > s$ and $1 \leq i \leq m_k$ we multiply (4.11) by $\phi_i^k(x')$ and integrate over Ω'. Defining p_i^k by

$$p_i^k(x_1) = \int_{\Omega'} u(x_1,x')\phi_i^k(x')dx' \in I\!R^p$$

we obtain after two integration by parts:

$$a^{\pm}p_i^{k''} + b^{\pm}p_i^{k'} + (a^{\pm}\omega_k + c^{\pm} - \lambda)p_i^k = 0.$$

Here the prime denotes derivation with respect to x_1. Since p_i^k is bounded we may view it as a tempered distribution, thus we may apply the Fourier transform with respect to x_1. Since $E_k^{\pm}(\xi,\lambda)$ is invertible we obtain that $\hat{p}_i^k$ is identically zero, thus by the inversion theorem p_i^k also is, a contradiction.

Proof of Proposition 4.6
We prove that $\alpha(L_2 - \lambda)$ is finite by showing that the unit ball in the null space $N(L_2 - \lambda)$ is compact in $\tilde{E}_s$. Consider a sequence $u_k \in N(L_2 - \lambda)$, with $|u_k|_{2+\delta,\Omega} = 1$. Then we can find a subsequence (still denoted by u_k) which converges to some function u_0 in the space $C^2(A)^p$ for any compact subset A of $\bar{\Omega}$. Note that $u_0 \in C^{2+\delta}(\bar{\Omega}, I\!R^p)$. We are going to show that the convergence is uniform on $\bar{\Omega}$. If this were not the case, we would have (again for some subsequence)

$$|u_k(x^k) - u_0(x^k)| \geq \delta > 0$$

for some $x^k \in \Omega$. Because of the uniform convergence on compact sets we have

$$|x^k| \to \infty.$$

Consider now the following shifted functions:

$$w_k(x) = u_k(x_1 + x_1^k, x') - u_0(x_1 + x_1^k, x').$$

Note that the sequence w_k is bounded in $C^{2+\delta}(\bar{\Omega}, I\!R^p)$, and that:

$$|w_k(0, x_2^k, .., x_n^k)| \geq \delta.$$

Thus we can find a subsequence which converges to some $w_0 \in C^{2+\delta}(\bar{\Omega}, I\!R^p)$, in $C^2(A)^p$ for any compact subset A of $\bar{\Omega}$. Passing to the limit $k \to \infty$ in the relations:

$$a(x_1 + x_1^k)\Delta w_k(x) + b(x_1 + x_1^k)\frac{\partial w_k}{\partial x_1}(x) + c(x_1 + x_1^k)w_k(x) = \lambda w_k(x) \quad x \in \Omega,$$
$$w_k(x) = 0 \quad x \in \partial\Omega,$$

we obtain that w_0 is a nontrivial solution to the problem (4.11). We now show that $w_0 \in R(Q_s)$. Applying Remark 4.3 to u_k we see that u_0, and therefore $u_k - u_0$, belong to $R(Q_s)$. Clearly any shift of an element of $R(Q_s)$ remains in $R(Q_s)$, thus $w_k, w_0 \in R(Q_s)$. This contradicts the conclusion of Lemma 4.8, thus we have shown that u_k converges to u_0 uniformly on $\bar{\Omega}$. It then follows from Schauder's estimates that the convergence also holds in $C^{2+\delta}(\bar{\Omega}, I\!R^p)$, and this completes the proof of Proposition 4.6.

Proof of Proposition 4.7

Pick a sequence $f_n = (L_2 - \lambda)u_n$ in $R(L_2 - \lambda) = R(L) \cap R(Q_s)$, converging to f in $C^\delta(\bar{\Omega}, I\!R^p)$. Because Q_s is a bounded operator we have $f \in R(Q_s)$. From Proposition 4.6 it follows that $N(L_2 - \lambda)$ has a direct complement W in $R(Q_s)$, thus we may write

$$u_n = v_n + w_n, \quad v_n \in N(L_2 - \lambda), w_n \in W.$$

If w_n is bounded in $C^{2+\delta}(\bar{\Omega}, I\!R^p)$, then as in the previous proof we can obtain a subsequence (still denoted w_n) which converges to some $w_0 \in E$ in $C^2(A, I\!R^p)$ for any compact subset A of $\bar{\Omega}$. Thus $f_n = (L_2 - \lambda)w_n$ converges to $(L_2 - \lambda)w_0$, and $f \in R(L_2 - \lambda)$. If w_n is not bounded in $C^{2+\delta}(\bar{\Omega}, I\!R^p)$, then let us set:

$$\bar{w}_n = \frac{w_n}{|w_n|_{2+\delta,\bar{\Omega}}}, \quad \bar{f}_n = \frac{f_n}{|w_n|_{2+\delta,\bar{\Omega}}}.$$

As above we extract a subsequence of $\bar{w}_n$ converging in C^2 on compact sets to some $w_0 \in W$. Then $(L_2 - \lambda)\bar{w}_n = \bar{f}_n$ converges to 0, thus $(L_2 - \lambda)w_0 = 0$, a contradiction. This completes the proof of the Proposition.

Thus we have proved the following theorem:

Theorem 4.9
Consider the operator

$$L: \quad C^{\delta}(\bar{\Omega}, I\!R^p) \longrightarrow C^{\delta}(\bar{\Omega}, I\!R^p), \quad D(L) = \{u \in C^{2+\delta}(\bar{\Omega}, I\!R^p): \; u_{|S} = 0\}$$

defined by

$$Lu = a(x)\Delta u + \sum_{j=1}^{n} b_j(x)\frac{\partial u}{\partial x_j} + c(x)u.$$

Assume that for any $j \geq 2$ we have $b_j(x) \to 0$ as $x_1 \to \pm\infty$, and assume Condition 1. Define the coefficients $\bar{a}(x_1)$, $\bar{b}(x_1)$, and $\bar{c}(x_1)$ by (4.2) (4.3), and let ω_k denote the eigenvalues of the Laplace operator in the section of the cylinder, with the boundary condition $u = 0$ on $\partial\Omega'$.

Then the operator L is Fredholm if and only if the operators

$$L^k: \quad C^{2+\delta}(I\!R, I\!R^p) \longrightarrow C^{\delta}(I\!R, I\!R^p)$$

defined by

$$L^k v = \bar{a}(x_1)v'' + \bar{b}(x_1)v' + (\bar{c}(x_1) + \bar{a}(x_1)\omega_k)v$$

are Fredholm for all k. This occurs if and only if for any integer k and all $\xi \in I\!R$, the matrices

$$E_k(\xi) := -a^{\pm}\xi^2 + ib_1^{\pm}\xi + c^{\pm} + a^{\pm}\omega_k$$

are invertible. In this case only finitely many of the indices $ind(L^k)$ are different from zero and

$$ind(L) = \sum_{k=1}^{\infty} m_k \; ind(L^k),$$

where m_k is the multiplicity of the eigenvalue ω_k.

5. Appendix

1: A lemma on the index of a direct sum
Let X be a Banach space, and $L: X \to X$ an unbounded normally solvable operator with domain $D(L) = E$. Let P, Q be two bounded projectors acting in X, with $P + Q = I_X$. Assume that for any $x \in E$:

$$Px \in E, \quad \text{and} \quad LPx = PLx.$$

Define two (unbounded) operators L_1, L_2 by taking restrictions of L:

$$L_1: R(P) \to R(P), \quad L_2: R(Q) \to R(Q),$$
$$D(L_1) = R(P) \cap E, \quad D(L_2) = R(Q) \cap E.$$

Then L_1 and L_2 are normally solvable, and:

$$ind(L) = ind(L_1) + ind(L_2).$$

2: Invertibility of the operator $L - \mu$

Let the operator L be given by (1.1). We show that for real, positive, and sufficiently large μ, the operator $L - \mu$ is Fredholm, and the dimension of its kernel and the codimension of its image equal zero.

The Fredholm property of the operator $L - \mu$ follows from Theorem 2.1 and from the remark after the theorem. Its index is zero because there exists a homotopy of this operator to the operator $\Delta - \mu$ in the class of Fredholm operators [18].

It remains to show that the problem

$$Lu - \mu u = 0, \ \ u|_{\partial\Omega} = 0$$

has no nontrivial solution. To this end we introduce a positive function $\phi(x_1)$ such that $\phi(x_1) = exp(-\sigma|x_1|)$ for some positive σ and for $|x_1| \geq 1$. We multiply the last equation by ϕa^{-1}, take the inner product with u, and integrate over Ω. After some standard computations we obtain:

$$0 \leq \sum_{i=1}^{p} \int_\Omega (-\phi + \frac{\epsilon}{2}|\phi^{'}| + M_1\frac{\epsilon}{2}\phi)|\nabla u_i|^2 dx + \int_\Omega (\frac{1}{2\epsilon}|\phi^{'}| + \frac{M_2}{2\epsilon}\phi - \mu k_1 \phi)|u|^2 dx. \qquad (5.1)$$

Here we used the fact that the matrix a^{-1} is positive definite:

$$(a^{-1}(x)u, u) \geq k_1(u, u), \ \ x \in \bar{\Omega}, \ \ u \in I\!R^p,$$

and inequalities of the type

$$|\phi \frac{\partial u_j}{\partial x_i} u_k| \leq \frac{\epsilon}{2}\phi |\frac{\partial u_j}{\partial x_i}|^2 + \frac{1}{2\epsilon}\phi|u_k|^2.$$

The constants M_1 and M_2 are positive and independent of $u(x)$ and $\phi(x_1)$, $\epsilon > 0$ can be taken arbitrary. For ϵ sufficiently small and μ sufficiently large, (5.1) can hold only for $u \equiv 0$.

References

[1] P. Bolley, T.L. Pham, Propriétés d'indice en théorie höldérienne pour des opérateurs différentiels elliptiques dans $I\!R^n$, *J. Math. Pures Appl.* 72 (1993), No. 1, pp. 105-119.

[2] B. Boos, D.D. Bleecker, Topology and Analysis. The Atiyah-Singer formula and gauge-theoretic physics, Springer-Verlag, Universitext, Berlin, 1985.

[3] M. Dauge, Elliptic boundary value problems in corner domains, *Lecture Notes Math.* 1341 (1988), Springer-Verlag, Berlin.

[4] I. Gohberg, S. Goldberg, M.A. Kaashoek, Classes of linear operators, Vol.1, Operator theory: Advances and applications, Birkhauser, 1990.

[5] I. Gohberg, M. Krein, The basic propositions on defect numbers and indices of linear operators, *AMS Transl.* **13**, No. 2 (1960), 185-264.

[6] D. Henry, Geometric theory of semilinear parabolic equations, Lecture Notes Math. 840, Springer-Verlag, Berlin, 1981.

[7] R. Horn, C. Johnson, *Matrix Analysis*, Cambridge Univ. Press, Cambridge, 1985.

[8] R.B. Lockhart, Fredholm properties of a class of elliptic operators on noncompact manifolds, *Duke Math. J.* 48 (1981), No. 1, pp. 289-312.

[9] R.B. Lockhart, R.C. McOwen, On elliptic systems in $I\!R^n$, *Acta Math.* **150** (1983), pp. 125-135.

[10] R.B. Lockhart, R.C. McOwen, Correction to "On elliptic systems in $I\!R^n$", *Acta Math.* **153** (1984), pp. 303-304.

[11] E.M. Mukhamadiev, *Normal solvability and the Noethericity of elliptic operators in spaces of functions on $I\!R^n$. Part I*, J. Soviet Math. **25** (1984), pp. 884-901.

[12] L. Nirenberg, H.F. Walker, *The null-spaces of elliptic partial differential operators in $I\!R^n$*, J. Math. Anal. Appl. **42** (1973), pp. 271-301.

[13] V.S. Rabinovich, *The Fredholm property of general boundary value problems on noncompact manifolds, and limit operators*, Russian Acad. Sci. Dokl. Math. **46** (1993), No. 1, pp. 53-58.

[14] B. Rowley, *An index formula for elliptic systems in the plane*, Trans. AMS **349** (1997), pp. 3149-3179.

[15] N. Shimakura, Partial Differential Operators of Elliptic type, Transl. of Math. Monographs, Vol. 99, AMS, Providence, 1991.

[16] A.I. Volpert, On the index and normal solvability of boundary value problems for elliptic systems in the plane, *Trudy Mosk. Mat. Obstsestva* (1961), 41-87.

[17] A.I. Volpert, V.A. Volpert, Construction of the Leray-Schauder degree for elliptic operators in unbounded domains, *Ann. Inst. H. Poincaré* **11**, No. 3 (1994), 245-273.

[18] A.I. Volpert, V.A. Volpert, J.F. Collet Topological degree for elliptic operators in unbounded cylinders, To appear in *Adv. Diff. Eq.*

[19] A.I. Volpert, Vi.A. Volpert, Vl.A. Volpert, Travelling wave solutions of parabolic systems, Transl. of Math. Monographs, Vol. 140, AMS, Providence, 1994.

List of symbols

$a, a_0, a^{\pm}, \bar{a}, A, A_{\infty}, A_{\pm}, A_+, A_-$
$b, b^{\pm}, b_0, b_i, b_i^{\pm}, b_1^+, b_1^-, b_k^{\pm}, \bar{b}, B$
$c, c^{\pm}, C, C^2, C^p, C^0$
$C^{l+\delta}(\bar{\Omega}, \mathbb{R}^p), C^{l+\delta}(\bar{\Omega}', \mathbb{R}^p), C^{l+\delta}(\bar{\Omega}, \mathbb{C}^p), C^{l+\delta}(\mathbb{R}, \mathbb{R}^p), C_b^l(\mathbb{R}, \mathbb{R}^p)$
$D_j^{\pm}$
$E_{\pm}, E_+, E_-, E', E_s, E'_s, \bar{E}'_s$
$f, f_k, f_1, f_2, f_n, \bar{f}_n, F_i$
g, G_i
h, H
i, I, I_1, I_2, I_p, I_X
j
k, K
$L, L^{\pm}, L_{\xi}^{\pm}, L_0, L_1, L_2, L^k, L_{\tau}, L^{\tau}, L^0, L^1, \bar{L}$
m_k, M
$n, n_j^+, n_j^-, N, N_1, N_2$
$p, p_i, p_i^k, P, P_s, P'_s$
Q, Q_s, Q'_s
r, R, R^p, R^{n-1}
s, s_1, s_2, S
$t, t_1, t_2, T, \bar{T}, T_{\tau}^{\pm}$
$u, u_0, u_1, u_k, u_n, u_p, u', u'', U, U^*$
v, v_1, v_2, v_n
$w_k, w_n, \bar{w}_n$
$x, x_i, x_1, x_p, x^k, x_2^k, x_n^k, x'$

α, α^k
$\beta, \beta^k, \beta_1^k, \beta_2^k$
$\gamma, \gamma_1^k, \gamma_2^k, \Gamma$
$\delta, \Delta, \Delta', \underline{\Delta}'$
$\kappa, \kappa^+, \kappa^-$
λ, Λ
μ
π, π_k, π'_k
ϕ, ϕ_i^k, Φ
ψ
τ, τ_0
ξ

$\omega_1, \omega_2, \omega_j, \omega_k, \Omega, \Omega^{'}$

$\sum_{i=1}^{n}$, ∂, ind, ind_k^0, $\cap$, $\otimes$, $\oslash$, $\in$, $>$, dim, $\rightarrow$, det, $sign$, $:=$, $\int$, $\|\ \|$, $|\ |$, $\leq$